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Ideal Theory of Right Cones and Associated Rings

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Right cones are semigroups with a left cancellation law such that for any two elements a, b there exists an element c with $b = ac$ or $a = bc$. Valuation rings, cones of ordered or left ordered groups, semigroups of ordinal numbers, and Hjelmslev rings are examples. The ideal theory of these semigroups is described in terms of prime and completely prime ideals, and a classification of prime segments is given that can be used to solve a problem raised by Skornyakov. The Archimedean case can be dealt with in a satisfactory way with the help of Hölder's theorem. Right cones of rank 1 are classified. We then consider the problem of constructing for a given right cone H a right chain ring R with the same right ideal and ideal structure as H. © 1998 Academic Press

INTRODUCTION

The right ideals of a ring R with identity are exactly the right ideals of the multiplicative semigroup $(R, \cdot)$ if and only if $aR \cup bR = bR$ or aR for any elements $a, b \in R$; i.e., $a = bc$ or $b = ac$ for some $c \in R$ and R is then a right chain ring. Since the maximal ideal $J = J(R)$ of such a ring is also the Jacobson radical, it follows that $ajR \subset aR$ for any $0 \neq a \in R$ and $j \in J$. Even though we are primarily interested in right chain rings, it appears reasonable to develop the theory as far as possible for semigroups with the two properties mentioned above (see Definition 1.1).

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A second reason for dealing with such a general setup that includes the cones of left ordered groups, the multiplicative semigroups of Hjelmslev rings, and semigroups of ordinal numbers or more general semigroups whose order is the natural right order (i.e., right holoids [S'e 79], [BT 87]; [BT 89]), is provided by the following observation: At least since Krull's construction of valuation rings in [Kr 32] with a given (commutative) ordered group as value group, many rings with given properties were constructed by considering groups or semigroups with related properties first, and then the desired rings were obtained by using something like localization of a semigroup ring or power series rings. We mention Jaffard's theorem [Jaf 53] that any lattice ordered abelian group G is a group of divisibility of a commutative integral domain, Neumann's construction [N 49] of a valuation ring by using generalized power series rings such that the principal fractional ideals form a group isomorphic to a given ordered group. In the noncommutative case there is, in addition, Cohn's construction [C 61] of a skew field into which the universal enveloping algebra of a Lie algebra can be embedded (see also [Da 70], [Li 95]). Cohn [C 85] constructs right chain domains for which the nonzero principal right ideals form a semigroup isomorphic to the right holoid O_I of all ordinal numbers less than ω^I, ω the order type of the natural numbers; see also Jategaonkar's construction [Jat 69] and the corresponding result for split holoids [BT 95], where wreath products of ordered groups are used. N. I. Dubrovin [Du 80] has constructed rank one valuation domains R with $R, J(R)$ and (0) as the only ideals with the help of a particular order defined by Smirnov [Sm 66] for an affine group over $\mathbb{R}$; and related results in [BS 95]. Finally, we mention Dubrovin's construction of rank one valuation domains with a prime ideal that is not completely prime and which shows that the group rings of certain left ordered groups can be embedded into skew fields, a partial solution of Malcev's problem [Du 93].

We develop the ideal theory for right cones H by considering the intersection of powers $I^n \neq \{\infty\}$ of an ideal $I \neq H$ of H and of $t^nH \neq \{\infty\}$ for $t \in J(H)$. These intersections are completely prime ideals, provided $\bigcap_n t^nH$ is an ideal (Proposition 1.11). This result is then used to prove Theorem 1.14, where prime segments $P_1 \supset P_2, P_i$ neighboring completely prime ideals of H are classified as either right invariant, simple, or exceptional, a classification that was known for chain domains and cones of groups. If H is the cone of an ordered group or the multiplicative semigroup of a chain domain whose skew field of quotients is finite dimensional over its center, then R has right invariant prime segments only [Gr 84]. The most complete results are obtained in the Archimedean case (Section 3), where it is shown that modulo the unit groups, we obtain either a subsemigroup of $(\mathbb{R} \cup \{\infty\}, +)$, $[0,1] \cup \{\infty\}$, or $[0,1]$ with appropriate operations, making use of Hölder's theorem (see [F 66]).

It may be of interest that Frege, trying to construct the real numbers at about the same time as Hölder in [Fr 03, Bd. II, p. 172], could not resolve the question of whether a left cone H in a group G with e as its only unit must also be a right cone; see [ADN 87] for such examples, which also exist in the wreath product of two infinite cyclic groups.

Of great interest is, of course, the rank 1 case with $\{\infty\}$ completely prime; these are in some way the building blocks of right cones. If the single prime segment is invariant, then H is Archimedean. In the other two cases no similarly complete result can be expected, but various possibilities are illustrated (see Section 3.1).

Skornyakov ([Sk 70]) studied left valuation semigroups, which are, in the language of this paper, left cones H with $U(H) = \{e\}$ and both cancellation laws. The question of whether the set of nilpotent elements in such a semigroup is an ideal was left open in [Sk 70]. We use the information obtained about prime ideals and the prime radical $P(H)$ of H to show in Theorem 2.2 that the set of nilpotent elements of H is an ideal if and only if $P(H)$ is completely prime.

In the second part of the paper we define and study associated right cones, i.e., right cones with the same right ideal structure and corresponding ideals and prime ideals; see Definition 4.1. It follows that a right cone H that is associated with a right chain domain must have $\{\infty\}$ as a completely prime ideal. Conversely, whether this condition is sufficient for H to be associated with a right cone H' such that $H' \setminus \{\infty\}$ is contained in a group (Section 4.1, Problem 1) or is associated with a right chain domain (Problem 2) remains an open question. We show, however, how a rank 3 right holoid that is neither split nor embeddable into a group is associated with a right chain domain (Example 4.2). In the process we use Proposition 4.3, which shows that a right cone H that is a subsemigroup of a group G and is compatible with an order of G is associated with a right chain domain.

1. CONES

1.1. *Definitions and Preliminary Results*

Let H be a semigroup with identity e. An element $a \in H$ with $a \cdot b = b \cdot a = a$ for all $b \in H$ is uniquely determined and denoted by $\{\infty\}$.

DEFINITION 1.1. A semigroup H with identity e is called a *right cone* if the following two conditions hold:

(i) For any two elements $a, b \in H$ there exists $c \in H$ with $a = bc$ or $b = ac$ (right chain condition).

(ii) If $ab = ac \neq \infty$ for elements $a, b, c \in H$, then $b = cu$ for a unit $u \in H$ (left cancellation).

H is called a *cone* provided H is a right and left cone.

In the definition of $a = \infty$ in H it is enough to require that $ab = a$ for all $b \in H$ if (i) is assumed. Indeed, $ba = ac = a$ or $a = bac = ba$ for some $c \in H$.

An element $u \in H$ is called a *unit* of H if there exists an element $v \in H$ with $uv = vu = e$; the set $U(H)$ of all units of H is a subgroup of H.

The condition (i) implies that for any two right ideals I_1, I_2 of H, either $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$, where a subset I of H is called a *right ideal* of H if $IH \subseteq H$. *Left ideals* and (two-sided) *ideals* are defined similarly. A right ideal $P \neq H$ of H is called *prime* if $aHb \subseteq P$ for $a, b \in H$ implies $a \in P$ or $b \in P$; if $ab \in P$ implies $a \in P$ or $b \in P$, then P is called *completely prime*.

EXAMPLE 1.2. If R is a ring with identity, and for any two elements $a, b \in R$ there exists $c \in R$ with $a = bc$ or $b = ac$, then we say that R is a *right chain ring*, and it follows that R is a right chain ring if and only if the multiplicative semigroup $(R, \cdot)$ is a right cone. Right chain rings whose non-units are two-sided zero-divisors are called *affine Hjelmslev rings* (originally introduced by Klingenberg; see [Lo 86]).

Condition (ii) holds for a right chain ring, since for a, b, c in R with $ab = ac \neq 0$, we have either $b = cu$ for u a unit in R, or we can assume $b = ct$ for some $t \in J$. Then $ac(1 - t) = 0$, which implies $ac = 0$, since $1 - t$ is a unit in R.

EXAMPLE 1.3. That condition (i) does not imply (ii) is illustrated by the semigroup $K = \{e, u, v\}$, with $u^2 = vu = u$ and $v^2 = uv = v$ (see [Sat 79] for other examples).

EXAMPLE 1.4. A semigroup P of a group G with $P \cup P^{-1} = G$ is a cone and defines a left (and right) order on G if also $P \cap P^{-1} = \{e\}$. The group G is ordered with P equal to the set of elements $\geq e$ if and only if the right and left orders defined by P agree, which is the case if and only if $gPg^{-1} = P$ for all $g \in G$.

EXAMPLE 1.5. If H_i are right cones without an element ∞ for $i = 1, 2$, then $H = \{(h_2, h_1) | h_i \in H_i\}$ is a right cone with $(h_2, h_1)(h'_2, h'_1)$ equal to $(h_2 h'_2, h'_1)$ if h'_2 has no inverse in H_2,and equal to $(h_2 h'_2, h_1 h'_1)$ if h'_2 has an inverse in H_2. The right cone O_2 of ordinal numbers $< \omega^2$ can be considered as such a product $O_2 = H$ for $H_1 = H_2 = \mathbb{N}$.

The element ∞ corresponds to the element 0 in the case where $H = (R, \cdot\,)$ and R is a right chain ring. If H does not contain such an element, we adjoin ∞ to H with $b \cdot \infty = \infty \cdot b = \infty$ for all $b \in H \cup \{\infty\}$ and assume from now on that H contains the element $\infty \neq e$.

We observe that the union $J(H) = J = \bigcup I$ of right ideals $I \neq H$ is again a right ideal $\neq H$ of H, which we occasionally call the (*Jacobson*) *radical* of H.

LEMMA 1.6. *Let H be a right cone. Then*:

(i) *J is an ideal.*

(ii) *All one-sided units in H are* (*two-sided*) *units*; *i.e., $uv = e$ implies $vu = e$ for elements $u, v \in H$.*

Proof. To show that $HJ \subseteq J$, we consider elements $j \in J$ and $r \in H$. If rj is not contained in J, then $r \notin J$ and $rjH = H = rH$. This implies $r = \infty \in J$, a contradiction.

To prove (ii). we assume that $uv = e$. If $vH = H$, then $vu' = e$ for some $u' \in H$ and $u = u'$ follows. Otherwise $v \in J$ and $uv \in J$ by (i), a contradiction. ▮

It follows from Lemma 1.6 that $U(H) = H \setminus J$ is the *group of units* of H. The following example was suggested by M. Schröder. It shows that a semigroup with (i) of Definition 1.1 does not necessarily have a two-sided maximal right ideal.

EXAMPLE 1.7. Let H be the semigroup with identity e, generated by x and y with the single relation $xy = e$. Then $\{y^n H | n = 0, 1, 2, \ldots\}$ is the set of right ideals of H and the maximal right ideal yH is not two-sided.

We point out that the above terminology also covers also left ordered (resp. ordered) semigroups:

A semigroup S is called *left ordered* if S is totally ordered by the relation $\leq$ and $a \leq b$ implies $ca \leq cb$ for $a, b, c \in S$.

For right cones we have the following result:

LEMMA 1.8. *Let H be a right cone.*

(a) *The following conditions are equivalent*:

(i) *H is left ordered by the relation $a \leq b$, which holds if and only if $b = ac$ for some $c \in H$.*

(ii) $U(H) = \{e\}$

(b) *The following conditions are equivalent*:

(i) *H is left and right ordered by the relation $a \leq b$, which holds if and only if $b = ac$ for some $c \in H$.*

(ii) *$U(H) = \{e\}$, and $Ha \subseteq aH$ for all $a \in H$; i.e., H is right invariant.*

Proof. (a) If $e \neq u \in U(H)$, then $e \leq u \leq e$, a contradiction of the assumption that H is left ordered by $\leq$, and $U(H) = \{e\}$ follows.

If, conversely, $U(H) = \{e\}$ and a, b are elements in H, then exactly one of the following possibilities occurs: $a = b$, $a = bc$ for $c \neq e$ and $b \neq \infty$, which means $b < a$ or $b = ad$ for $a \neq \infty$, $d \neq e$, and this means $a < b$. That $a \leq b$ implies $ca \leq cb$ for $a, b, c, \in H$ follows immediately.

To prove (b) it remains to show that the left order discussed under (a) is also a right order if and only if H is right invariant. If we assume that $\leq$ also defines a right order, then $e \leq b$ implies $a \leq ba$ for $a, b \in H$, and hence $ba = ac$ for some $c \in H$; i.e., H is right invariant. Conversely, if H is right invariant and $a \leq b$ for $a, b \in H$, then $b = ac$ and $ad \leq bd = acd = adc'$ for $cd = dc'$ and some $c' \in H$. ▮

EXAMPLE 1.9. Right cones H that satisfy one of the equivalent conditions under Lemma 1.8(b) with $\{\infty\}$ a completely prime ideal are called *right holoids*. The semigroup $O_I = (\{\alpha | \alpha < \omega^I\}, +)$ of ordinal numbers less than a power of ω, the order type of $\mathbb{N}$, is an example of a right holoid (see also [S'e 79], [BT 87]).

1.2. *Prime Ideals in Right Cones*

We characterize prime and completely prime ideals in a right cone H.

LEMMA 1.10. *Let H be a right cone.*

(i) *An ideal $I \neq H$ of H is completely prime if and only if $x^2 \in I$ implies $x \in I$.*

(ii) *A right ideal $I \neq H$ of H is prime if and only if $xHx \subseteq I$ implies $x \in I$.*

Proof. (i) The condition of Lemma 1.10 is clearly necessary. Conversely, assume that the condition holds and $xy \in I$ for $x, y \in H$. If $x = yc$, then $x^2 = xyc \in I$ and $x \in I$ follows. If $y = xc$, then $(yx)^2 = y(xy)x \in I$; hence $yx \in I$ and $y \in I$, as in the first step.

(ii) We must show that under the condition in (ii), the assumption $aHb \subseteq I$ implies $a \in I$ or $b \in I$. If $a = bc$, we have $aHa \subseteq I$ and $a \in I$ follows. If $b = ac$, then $bHb = acHb \subseteq aHb \subseteq I$ and $b \in I$ follows. ▮

PROPOSITION 1.11. *Let H be a right cone and $I \neq H$ an ideal.*

(i) *Then $\bigcap_{n \in \mathbb{N}} I^n = P$ is a completely prime ideal of H if $I^n \neq \{\infty\}$ for all integers $n \in \mathbb{N}$.*

(ii) *Idempotent ideals $\neq H$, $\{\infty\}$ are completely prime.*

(iii) *If* $t \in J = J(H)$, *then* $\bigcap_{n \in \mathbb{N}} t^n H = P$ *is a prime right ideal if* $t^n H \neq \{\infty\}$ *for all* n. *If* P *is, in addition, an ideal, then* P *is completely prime.*

Proof. (i) If $I = I^2$, then $P = I$, and we assume that $x^2 \in I, x \notin I$ for some $x \in H$. Then $I \subseteq xJ$ and $I = I^2 \subseteq xI \subseteq x^2 J \subset x^2 H \subseteq I$, a contradiction that shows, using Lemma 1.10, that I is completely prime if $I = I^2$.

If $P = I^n$ for some $n = 2, 3, \ldots,$ then $P = I^n = (I^n)^2$ is completely prime by the previous argument.

Finally, if $P \subset I^n$ for all n and $x \notin P$, it follows that there exists an integer n with $I^n \subseteq xH$ and then $P \subset I^{2n} \subseteq xI^n \subseteq x^2 H$, and hence $x^2 \notin P$; applying Lemma 1.10 again, we conclude that P is a completely prime ideal.

(ii) follows directly from (i).

(iii) Since $t \in J$, it follows that $t^{n+1}H \subset t^n H$ by condition (ii) (Definition 1.1) and the assumption $t^n \neq \infty$, and hence $P \subset t^n H$ for all integers n. If $x \notin P$, then there exists a natural number n with $t^n H \subseteq xH$; hence $t^{2n} H \subseteq t^n xH \subseteq xHxH$, and $xHx \not\subseteq P$ follows, which proves that P is a prime right ideal. If we assume, in addition, that P is an ideal with $x \notin P$, then there exists an n with $t^n = xa$ for some $a \in H$. Therefore $t^{2n} = xaxa$, and if $ax \in xH$, we have $t^{2n} = x^2 b$ for some $b \in H$ and $x^2 \notin P$ follows. Otherwise, $axj = x$ for some $j \in J$ and $xj \notin P$, since $x \notin P$ and P is assumed to be an ideal. Therefore, there exists an m with $xjr = t^m$ for some $r \in H$, and $x^2 r = x(axj)r = (xa)(xjr) = t^n \cdot t^m$ implies $x^2 \notin P$. ∎

Next we investigate prime ideals which are not completely prime and which we call *exceptional*. The first examples of such ideals were given by Dubrovin in [Du 84], where it is shown that the group generated by x, y with the relation $y = xy^2 x$ contains a cone P with a prime ideal that is not completely prime. That such prime ideals also exist in chain rings was shown in [Du 93].

Let Q be an exceptional prime ideal. By Proposition 1.11 (ii) it cannot be idempotent, unless $Q = \{\infty\}$. We set P as the intersection of all completely prime ideals containing Q. Obviously P exists, since J is completely prime. By Proposition 1.11 (iii) the intersection $\bigcap_{n \in \mathbb{N}} P^n$ is completely prime and contains Q. So P must be idempotent. Furthermore, since Q is prime and P is minimal completely prime over Q, we conclude with Proposition 1.11 (i) that there are no ideals between P and Q. Thus we have proved the following:

LEMMA 1.12 (Pairing Lemma). *Let* H *be a right cone and* Q *be an exceptional prime ideal. Then* $Q = \{\infty\}$ *or* $Q \supset Q^2$ *holds. Furthermore, there exists a completely prime ideal* P *minimal over* Q. *Finally,* P *is idempotent and there is no ideal between* P *and* Q.

It is obvious by Lemma 1.10 (i) that for an exceptional prime ideal Q there exist elements $a \notin Q$ with $a^2 \in Q$. We say that an element $a \in H$ is Q-nilpotent if $a^n \in Q$ for some $n \in \mathbb{N}$. The next corollary presents some further information on exceptional prime ideals.

LEMMA 1.13. *Let H be a right cone, Q an exceptional prime ideal, and P the completely prime ideal minimal over Q. Then we have*

(i) *There exist elements $a \in P \setminus Q$ satisfying $a^2 \notin Q$.*

(ii) *There exist elements $a \in P \setminus Q$ satisfying $\bigcap_{n \in \mathbb{N}} a^n H \supset Q$. In particular, we have elements in $P \setminus Q$ that are not Q-nilpotent.*

Proof. (i) Follows from the following statement: Let P be a right ideal and Q an ideal with $P \supseteq Q$ and $a^2 \in Q$ for all $a \in P$. Then $P^3 \subseteq Q$.

To prove this let $a, b, c \in P$. We want to show that $x = abc \in Q$. If $b = ar$, we have $x = a^2rc \in Q$. Otherwise, $a = br$ for some $r \in H$ and $x = brbc$. If $bc = brt$ for $t \in H$ we are done; otherwise $br = bct$ and $x = bctbc$. We now compare ctb with b and $x = bbvc$ if $ctb = bv$ for some $v \in H$. We are left with the possibility that $ctbv = b$. In this case $b = (ct)bv = (ct)^2bv^2$ in in Q and $x \in Q$ follows.

If (i) does not hold, then $P^3 \subseteq Q$ implies $P \subseteq Q$, since Q is prime. Thus there exist elements $P \setminus Q$ whose squares are not in Q.

(ii) By (i) there exists $a \in P \setminus Q$ with $a^2 \notin Q$. We will show that under the assumption that any $x \in P \setminus Q$ is Q-nilpotent, we conclude that $Ha^2 \subseteq aH$. In this case $Ha^2H \subseteq aH$ would be an ideal properly between P and Q, which is impossible.

Now to prove $Ha^2 \subseteq aH$. We are done, if for every $r \in H$ there is an $s \in H$ with $ra^2 = as$. So assume $ra^2s = a$ with $r, s \in H$; hence $a = raas = ra(ra^2s)s = \cdots = (ra)^n as^n$; thus for n large enough we would obtain $(ra)^n \in Q$ and so $a \in Q$, a contradiction.

As a consequence, there exist elements a that are not Q-nilpotent. Finally, by Proposition 1.11 (iii) the intersection $\bigcap_{n \in \mathbb{N}} a^n H$ is larger than Q; otherwise Q would be completely prime. ∎

1.3. *Prime Segments in Right Cones*

A *prime segment* $P_1 \supset P_2$ of H consists of two neighboring completely prime ideals of H, i.e., $P_1 \neq P_2$, but no further completely prime ideal exists between P_1 and P_2. Special cases of the next result have appeared previously; we mention [BT 76], where rank one chain domains are classified, and [Du 96], where cones in groups are discussed. Related results and particular examples can be found in [BS 95].

THEOREM 1.14. *Let H be a right cone and let $P_1 \supset P_2$ be a prime segment of H. Then exactly one of the following alternatives occurs:*

(a) *The prime segment* $P_1 \supset P_2$ *is* (*locally*) *right invariant*; *i.e.*, $P_1 a \subseteq aP_1$ *for all* $a \in P_1 \setminus P_2$.

(b) *The prime segment* $P_1 \supset P_2$ *is simple*; *i.e.*, *there are no further ideals between* P_1 *and* P_2.

(c) *The prime segment* $P_1 \supset P_2$ *is exceptional*; *i.e.*, *there exists a prime ideal* Q *with* $P_1 \supset Q \supset P_2$ *and no further ideal between* P_1 *and* Q. *The intersection* $\bigcap_{n \in \mathbb{N}} Q^n$ *equals* P_2.

Proof. We consider the union L of ideals properly contained in P_1. If $L = P_2$, the prime segment is simple, Case (b).

If L is properly between P_1 and P_2 and if P_1 is idempotent, let $A \supset L$ and $B \supset L$ be ideals of H, then $A \supseteq P_1$ and $B \supseteq P_1$. Then $AB \supseteq P_1 P_1 = P_1$ and L is an exceptional prime ideal; we are in Case (c). The rest follows from Lemma 1.12.

To complete the proof it must be shown that the segment $P_1 \supset P_2$ is right invariant if P_1 is not idempotent or if $L = P_1$. Under these conditions there exists for a given element $a \in P_1$ an ideal $I \subseteq P_1$ with $a \in I$ and $\bigcap_{n \in \mathbb{N}} I^n = P_2$, where we again use Proposition 1.11 (i).

Let $a \in P_1 \setminus P_2$ and $p \in P_1$. If $pa = ap'$ for $p' \in P_1$, we are done. *If* $pa = as$ for $s \notin P_1$, then $p \in I$ for a suitable ideal I and $\bigcap_{n \in \mathbb{N}} I^n = P_2$. Then there exists an integer n with $p^n \in I^n \subseteq a^2 H$ since $a^2 H \supset P_2$. Therefore, $p^n a = a^2 z$ for some $z \in H$, and $a^2 z = p^n a = as^n$ implies $azH = s^n H$, where we use condition (ii) in Definition 1.1 and the fact that $as^n \neq \infty$, since $as^n \notin P_2$. However, $az \in P_1$ and $s^n \notin P_1$, a contradiction.

We are left with the case where $paj = a$ for $j \in J$ and $p^n aj^n = a$ follows for all natural numbers. Then with $I^n \subset aH$ for n large enough, we obtain the contradiction $a = p^n aj^n \in I^n$, since $p \in I$.

Finally, we observe that the three alternatives in the theorem are mutually exclusive. This is obvious, except for the possibility that a segment satisfies (a) and (c). In this case let $a \in P_1 \setminus Q$, and $P_1 a P_1 = P_1(HaH)P_1 \not\subseteq Q$, since Q is prime. However, $P_1 a P_1 \subseteq aP_1^2 \subseteq aH$ and $P_1 a P_1 \neq P_1$, unless $aH = P_1$. This is not possible, since in that case $Q \subset P_1^2 = a^2 H \subset aH$, violating the assumption that (c) holds. ▮

COROLLARY 1.15. *Let* H *be a right cone and let* $P_1 \supset P_2$ *be a right invariant segment*. *Then* $\bigcap_{n \in \mathbb{N}} t^n H = P_2$ *for every* $t \in P_1 \setminus P_2$.

Proof. We use Proposition 1.11 (iii), and it is enough to show that $I = \bigcap_{n \in \mathbb{N}} t^n H$ is an ideal. If $I = P_2$ we are done; otherwise consider an element $x = t^{n+1} h \in I \setminus P_2$ for $h \in H$ and $r \in H$. Then rx is in P_2 and in I if r is in P_2. Otherwise $rx = (rt)t^n h = t^n r' h$ for a suitable element $r' \in H$, where we use $rt \in P_1 \setminus P_2$ and the right invariance of the segment $P_1 \supset P_2$. It follows that I is a two-sided and hence completely prime ideal. ▮

Every right cone H contains at least one completely prime ideal, namely, the Jacobson radical $J(H)$.

DEFINITION 1.16. We say that a right cone H has rank n if H has exactly $n+1$ completely prime ideals.

2. CHARACTERIZATION OF THE PRIME RADICAL AND A PROBLEM OF SKORNYAKOV

We apply the results of the previous section to gain insight into the *prime radical* $P(H)$ for H, which is defined as the intersection of all prime ideals of H and is itself a prime ideal. As in ring theory, we say that an element $a \in H$ is *strongly nilpotent* if for every sequence $\{a_n | n \in \mathbb{N}_0\}$ with $a_0 = a$, $a_{n+1} \in a_n H a_n$, there exists a natural number n with $a_n = \infty$; the element $a \in H$ is called *nilpotent* if there exists n with $a^n = \infty$.

PROPOSITION 2.1. *Let H be a right cone. The intersection $P(H) := \bigcap P_\lambda$ of all prime ideals P_λ of H is equal to the set of all strongly nilpotent elements in H.*

Proof. If an element $a \in H$ is not contained in $P(H)$, there exists a prime ideal P with $a \notin P_\lambda$, and therefore $aHa \not\subseteq P_\lambda$. Hence, there exists $a_1 \in aHa$, $a_1 \notin P_\lambda$. Repeating this argument, it follows that a is not strongly nilpotent. Conversely, we show (with the same argument as in ring theory) that an element $a \in H$ which is not strongly nilpotent is also not in $P(H)$. There exists a sequence $S = \{a_n | n = 0, 1, 2, \dots\}$ with $a_0 = a$ and $a_{n+1} \in a_n H a_n$ and $\infty \notin S$.

Therefore $S \cap \{\infty\} = \emptyset$, and there exists an ideal P in H maximal in the set of ideals I of H with $S \cap I = \emptyset$. If $A \supset P$, $B \supset P$ are ideals of H, then there exist $a_k \in A$, $a_t \in B$, and for $t \geq k$ it follows that $a_t \in A \cap B$. Then $a_{t+1} \in a_t H a_t \subseteq AB$, and hence $AB \subseteq P$ is impossible; i.e., P is a prime ideal with $a \notin P$, which proves the claim. ∎

Skornyakov in [Sk 70] leaves open the question of whether the set of nilpotent elements forms an ideal in a valuation semigroup (left cones H with $U(H) = \{e\}$ and both cancellation laws). The next result gives a precise answer, even in the case of right cones.

THEOREM 2.2. *Let H be a right cone. Then the set M of nilpotent elements of H is an ideal of H if and only if the prime radical $P(H)$ of H is completely prime.*

Proof. It follows from Proposition 2.1 that the elements in $P(H)$ are nilpotent. If $P(H)$ is completely prime, then every nilpotent element of H must be contained in $P(H)$ and $P(H) = M$ follows.

We now assume that $P(H) = Q$ is exceptional. Furthermore, we have $Q^n = \{\infty\}$ and let P be the minimal completely prime ideal of H containing Q (Lemma 1.12). Then there exist Q-nilpotent elements in $P \setminus Q$ that are then also nilpotent. If we assume that the set M of nilpotent elements is an ideal, then $M = P$, since there is no ideal properly between P and Q. However, $P \setminus Q$ contains elements that are not Q-nilpotent by Lemma 1.13 (ii) and so are not nilpotent, a contradiction. ∎

Summarizing our observations, we have obtained three alternatives for the prime radical. Parts (b) and (c) of the following lemma should be considered as a complement to Theorem 1.14.

LEMMA 2.3. *Let H be a right cone and $P(H) = P$ the prime radical of H. Then one of the following cases occurs*:

(a) *P is completely prime and $P = \{\infty\}$.*

(b) *$P \neq \{\infty\}$ is completely prime. Then $Pa \subseteq aP$ for all $a \in P$ with $aP \neq \{\infty\}$.*

(c) *P is exceptional. Then $P^n = \{\infty\}$ for some $n \in \mathbb{N}$.*

Proof. (b) By Theorem 2.2, P coincides in this case with the set of nilpotent elements. We now consider pa for $p, a \in P$. If $pa = as$ with $s \in P$, we are done. Otherwise, if $s \notin P$, then $\infty = p^n a = as^n$ for some n large enough and $s^n \notin P$. Hence $P \subseteq s^n H$ and $aP \subseteq as^n H = \{\infty\}$. Similarly, the case $pas = a$ cannot occur, since $\infty = p^n as^n = a$ contradicts our assumption $aP \neq \{\infty\}$.

(c) The intersection of all powers of P is a completely prime ideal, provided $P^k \neq \{\infty\}$ for all $k \in \mathbb{N}$ by Proposition 1.11 (i). So it follows that $P^n = \{\infty\}$ for some integer n. ∎

3. THE ARCHIMEDEAN CASE AND RANK 1 RIGHT CONES

The structure of right cones cannot be described completely, since this class contains, as was mentioned earlier, the class of cones of ordered groups and the class of multiplicative semigroups $(R, \cdot)$ for R a valuation ring. However, the situation improves if we add extra conditions.

We say that a right cone H is *Archimedean* if for any $a \neq \infty \neq b, a, b \in J(H)$ there exist natural numbers n, m with $a^n H \subseteq bH$, $b^m H \subseteq aH$.

LEMMA 3.1. *Let H be an Archimedean right cone. Then*:

(i) *H does not contain an exceptional prime*; *in particular, the radical of H is completely prime.*

(ii) *The rank of H is* ≤ 1.

(iii) *H is right invariant; i.e., $Ha \subseteq aH$ for all $a \in H$. Moreover, we have $U(H)a \subseteq aU(H)$ for all $a \in H$.*

(iv) *H satisfies the following (weak) right cancellation law: $ac = bc \neq \infty$, $a, b, c \in H$, implies $au = b$ for some $u \in U(H)$.*

(v) *$\overline{H} = H/U(H)$ exists, is again an Archimedean right cone, and $U(\overline{H}) = \{\bar{e}\}$.*

Proof. (i) H cannot contain an exceptional prime ideal Q, since otherwise by Lemma 1.13 (ii) elements $a \in J(H)$ with $\bigcap a^n H \supset Q$ would exist, contradicting the Archimedean property.

(ii) Since pairs of elements in different prime segments do not satisfy the Archimedean property, there exists at most two completely prime ideals. In the case where there are exactly two prime ideals, $P(H) = \{\infty\}$ holds.

(iii) If H has rank 0, then $P(H) = J$, which is the set of nilpotent elements. Any equation $ras = a$ with $r \in J$ or $s \in J$ leads to $r^n as^n = a = \infty$ for a sufficiently large n. This implies $Ha \subseteq aH$ for any $a \in H$.

If H has rank 1, then $J \supset \{\infty\} = P(H)$ is the single prime segment of H, and we consider the alternatives given by Proposition 1.14. By (i), Case (c) with an exceptional prime ideal cannot occur. To guarantee that the prime segment is not simple, we have to show that there exists a nontrivial ideal. If $J^2 \subset J$, we are done. Otherwise, we find aH such that $\infty \subset aH \subset J$ and prove that $Ha^2 \subseteq aH$ holds; hence $\{\infty\} \subset Ha^2H \subseteq aH$, and so Ha^2H is an ideal different from H, J and $\{\infty\}$.

Assume $ra^2s = a$ for some $r \in H, s \in J$ and $\infty \neq a \in J$; then $a = ra(ra^2s)s = (ra)^2as^2 = (ra)^n as^n$ follows, and so $(ra)^n H \supset aH$ for all natural numbers n, contradicting the Archimedean property.

Hence, by Theorem 1.14, the segment $J \supset \{\infty\}$ is right invariant and $Ja \subseteq aJ$ follows. If $uas = a$ for some $u \in U(H)$ and $s \in J(H)$, then $u^n as^n = a$ and $s^n = at$ for n big enough and some $t \in H$. Then $a = u^n as^n = (u^n a)(at) = atj'$ for some $j' \in J$, since $u^n a \in J$ and $Jat \subseteq atJ$. Hence, $Ua \subseteq aH$ and $Ha \subseteq aH$, which implies $Ua \subseteq aU$ by Definition 1.1 (ii).

(iv) Let $ac = bc \neq \infty$ and assume $ar = b$ for some $r \in H$. Thus $ac = arc$ and applying (ii) of Definition 1.1, we obtain $cH = rcH = r^n cH$. Since H is Archimedean, r has to be a unit and $aH = bH$ follows.

(v) We have $U(H)a \subseteq aU(H)$ for all $a \in H$ by (iii), and $\overline{H} = H/U(H)$ exists and is an Archimedean right cone. Since $U(\overline{H})$ is trivial, $\overline{H}$ is left and right cancellative by (iv); i.e., $ab = ac \neq \infty$ or $ba = ca \neq \infty$ implies $b = c$. ∎

It follows from Lemma 1.8 (b) that $\overline{H}$ is a left and right ordered semigroup with $a \leq b$ if and only if there exists $q \in \overline{H}$ with $aq = b$. We use this order in the proof of the next proposition.

PROPOSITION 3.2. *An Archimedean right cone H with $U(H) = \{e\}$ is commutative.*

Proof. We can assume $H \neq \{e, \infty\}$. If $J(H) = J$ is right principal, then $J = H \setminus \{e\}$ contains a smallest element a. Let $\infty \neq b \neq a$ be arbitrary in J; then an integer $k \geq 1$ exists with $a^k < b \leq a^{k+1}$. We have $b = a^k c$ for some $c \geq a$, so $c = aq$ for some $q \in H$, showing $b = a^{k+1}$, and the commutativity of H follows.

If J is not finitely generated as a right ideal, then there exist for every $x \in J$ elements y and $w \in J$ with $x = yw$; hence $z^2 \leq x$ for $z = \min\{y, w\}$. With the same arguments, $z \in J$ also exists, with $z^n \leq x (n \geq 2)$.

We observe that for any two elements $h_1, h_2 \in H$ with $h_1 d = h_2 \neq \infty$ and $d \in J$, there exists a natural number n with $d^n < h_1 \leq d^{n+1}$. Hence $h_1 \leq d^{n+1} \leq h_1 d = h_2$.

Next we want to show that the assumption $\infty \neq ba < ab$ for some elements $a, b \in H$ is impossible.

Case 1. There exists an element $w \in H$ with $ba < w < ab$, and let $bad = w \neq \infty$ for some element $d \in J$. We can find some $z \in J$ with $z^3 \leq d$, and there exists n with $z^n \leq ba < z^{n+1} < z^{n+2} \leq w < ab$. Finally, we find integers k, l satisfying $z^k \leq a < z^{k+1}$, $z^l \leq b < z^{l+1}$. It follows that ab as well as ba lie between z^{k+l} and z^{k+l+2}. Since $k + l \leq n$, we obtain a contradiction.

Case 2. There are no further elements between $ba < ab$. If $\infty \neq ab = bad$, then $J = dH$, a contradiction. Hence we can assume that $ab = \infty$. Let c be an element with $ac = ba < ab$. Hence $c < b$ and $ca < ba < \infty$. We apply Case 1 and obtain $ac = ca$; hence $ca = ac = ba$ and $c = b$, using the right cancellation law. ∎

The last proposition shows that for an Archimedean right cone H, the semigroup $\overline{H} = H/U(H)$ (see Lemma 3.1(v)) is a commutative Archimedean, naturally ordered semigroup. By a theorem of Hölder and Clifford (see [F 66], p. 228) we therefore obtain the following characterization of Archimedean right cones:

THEOREM 3.3. *Let H be an Archimedean right cone. Then one of the following alternatives holds*:

(i) *H has rank* 1; *hence $\{\infty\}$ is a completely prime ideal, and $\overline{H} = H/U(H)$ is isomorphic to a subsemigroup or* $(\mathbb{R} \cup \{\infty\}, +)$.

(ii) *H has rank* 0 *and either*

(a) *The ideal* $\{\infty\}$ *has an upper neighbor in the lattice of right ideals of H and $\overline{H}$ is isomorphic to a subsemigroup of* $[0,1] \cup \{\infty\}$ *with* $a \circ b = a + b$, *provided* $a + b \leq 1$ *and* ∞ *otherwise, or*

(b) *The ideal* $\{\infty\}$ *does not have an upper neighbor in the lattice of right ideals of H, and $\overline{H}$ is isomorphic to a subsemigroup of* $[0,1]$, *with* $a \circ b = a + b$, *provided* $a + b \leq 1$ *and* 1 *otherwise.*

In Case (ii)(b) the element $\infty \in \overline{H}$ corresponds to the 1 of the semigroup $[0,1]$.

COROLLARY 3.4. *The following conditions are equivalent for a right cone H:*

(a) *H is Archimedean.*

(b) *H is right invariant and has rank* ≤ 1, *and* $\{\infty\}$ *is a completely prime ideal in the case rank*$(H) = 1$.

Proof. Use Corollary 1.15 and Proposition 1.11 (iii). ∎

3.1. *Rank* 1 *right cones*

We consider in this section right cones H of rank 1 with $\{\infty\}$ a completely prime ideal. It follows from Theorem 1.14 that there are three alternatives for the prime segment $J \supset \{\infty\}$. If the prime segment $J \supset \{\infty\}$ is right invariant, then by Corollary 1.15 the right cone H is Archimedean. Hence by Lemma 3.1 (iii) and (iv) H is right invariant and $\overline{H} = H/U(H)$ exists. Finally, by Theorem 3.3 $\overline{H}$ is isomorphic to a subsemigroup or $(\mathbb{R} \cup \{\infty\}, +)$.

No similarly complete results exist for the simple and exceptional case. We therefore have to restrict ourselves to giving some examples.

We will construct rank 1 (right and left) cones H_1, H_2, and H_3, which have isomorphic lattices of right ideals, but are right invariant, nearly simple, and exceptional, respectively.

EXAMPLES 3.5. (1) We take for $H_1 = \{t^r | 0 \leq r \in \mathbb{R}\} \cup \{\infty\}$ the cone with $t^{r_1} t^{r_2} = t^{r_1 + r_2}$, which is isomorphic to $(\mathbb{R}^{\geq 0}, +)$, the nonnegative numbers under addition (with ∞ adjoint); $U(H_1) = \{t^0\} = \{1\}$.

(2) For $H_2 = \{t^r x^n | r \geq 0,\ n \in \mathbb{Z}\} \cup \{\infty\}$ we take the semigroup with $xt^r = t^{2r}x$ defining the multiplication. This is a rank 1 nearly simple cone with $t^r H_2$, $r \geq 0$ as the set of principal right ideals, but since $x^{-1} t^r H_2 = t^{r/2} H_2$, H_2 has no ideals besides H_2, $J(H_2) = \{t^r x^n | r > 0\}$ and $\{\infty\}$.

(3) Finally, we take for H_3 the subsemigroup $\{t^r u | 0 \leq r \in \mathbb{R},\ u \in U\}$ of the covering group of the group $SL(2, \mathbb{R})$ adjoining $\{\infty\}$, where

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| b, 0 < a \in \mathbb{R} \right\}$$

and $ut^r = ut^{2\pi k+\phi} = t^{2\pi k+\psi}u'$ with $0 \le \phi,\ \psi < 2\pi$ and

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$
$$= \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & a'^{-1} \end{pmatrix} \in SL(2,\mathbb{R})$$

for

$$u = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in U$$

and suitable

$$u' = \begin{pmatrix} a' & b' \\ 0 & a'^{-1} \end{pmatrix} \in U.$$

In that case, $t^{\pi}H_3 = Q$ is a prime ideal that is not completely prime and H_3 is an exceptional rank 1 cone ([Du 94]).

4. ASSOCIATED CONES AND RINGS

We consider right cones that have isomorphic lattices of right ideals so that ideals correspond to ideals and prime ideals to prime ideals, a relationship that holds between a valuation right and its associated semigroup of values. It follows from the Malcev–Neumann construction ([N 49]) that the positive cone P of any ordered group is associated with a chain domain R, but it is not known whether the positive cone of an arbitrary left ordered group is associated with a right chain domain. Similarly, it was proved in [BT 95] that split holoids are associated with right chain domains, but again, it is not known whether an arbitrary right holoid can be associated with a suitable right chain domain.

4.1. *Associations*

Let H_i, $i = 1, 2$ be right cones and let $\mathscr{I}(H_i)$ denote the set of right H_i-ideals. We define an order relation $\le$ on $\mathscr{I}(H_i)$ through

$$A \le B \Leftrightarrow B \subseteq A,$$

where A, B are right ideals in H_i.

DEFINITION 4.1. We say that H_1 and H_2 are *α-associated* if there exists an order p preserving (bijective) correspondence $\alpha\colon (\mathscr{I}(H_1), \le) \to (\mathscr{I}(H_2), \le)$ such that ideals are mapped to ideals, prime ideals to prime ideals, and completely prime ideals to completely prime ideals. We also call the isotone bijection α an *association*.

It follows that associated right cones have the same rank, that prime segments correspond to prime segments, and that corresponding prime segments are of the same type (by Theorem 1.14).

However, if H_1 and H_2 are α-associated right cones, it is not necessarily true that $\alpha(AB) = \alpha(A)\alpha(B)$ for ideals A and B of H_1. For example, consider $H_1 = H_2$, as in Example 3.5(1) and let $\alpha(t^r H_1) = t^{r^2} H_1$. The examples in 3.5 also show that there can exist an order preserving bijection α between $(\mathscr{T}(H_1), \leq)$ and $(\mathscr{T}(H_2), \leq)$ for right cones H_1 and H_2, so that ideals do not necessarily correspond to ideals, and prime ideals do not necessarily correspond to prime ideals.

The following two problems can be considered as a generalization of the problem of constructing for every cone of an ordered group an associated valuation ring.

Problem 1. Which right cones $(H, \cdot)$ are associated with a right cone $(P \cup \{\infty\}, \cdot)$ with $P \subseteq G$ and $(G, \cdot)$ a group?

Even more challenging is the next question:

Problem 2. Which right cones $(H, \cdot)$ are associated with a right cone $(R, \cdot)$ with R a right chain ring?

Obviously we must assume in Problem 1 that $\{\infty\}$ is a completely prime ideal of H. We have only partial answers to either of these problems. We mentioned above that it is unknown whether every cone P of a right ordered group (G, P) is associated with a right cone $(R^*, \cdot)$ with R a chain domain. This is related to Malcev's problem, which asks whether the group ring $\mathbb{Q}G$ for G right ordered is embeddable into a skew field. However, Dubrovin (see [Du 94]) has shown that for right ordered groups (G, P) satisfying certain conditions, there exists a skew field $F \supseteq G$ containing a chain domain R so that $R \cap G = P$, and every element $0 \neq r \in R$ can be written as $r = vp = qu$ for $p, q \in P$, $v, u \in U(R)$ and $PpP = PqP$; it follows then that P and $(R^*, \cdot)$ are associated (see also [BS 95] for the case where $\mathbb{Q}G$ is a right Ore domain). This result can be applied to Example 3.5(3). We point out that the right cone $H = O_I = \{\alpha \mid \alpha < \omega^I\}$ is not embeddable into group if $I \geq 2$, but is associated with a right cone P contained in a group, as well as with a right chain domain.

On the other hand, $H \setminus \{\infty\}$ is embeddable into a group for a right cone H if and only if $\{\infty\}$ is a completely prime ideal and H satisfies both (strong) cancellation laws or, equivalently, $ax = a \neq \infty$ or $xa = a$ implies $x = e$ for $a, x \in H$.

With respect to Problem 2, we mention that right holoids H of rank less than 3 and right noetherian holoids are associated with right chain domains ([Jat 69], [C 85], [BBF 77]). In the next example we consider a not right noetherian rank 3 holoid which is not a split holoid, and we use some new arguments and variations of some results obtained in [BT 95].

EXAMPLE 4.2. Let H be a right holoid generated by x, y, z, zx^{-n}, with $xy = y$, $xz = z$ and $yz = zx$. The elements of H are of the form $z^n y^m x^k$ for $0 \leq n, m, k$ or of the form $z^n x^{-k}$ for $n, k > 0$. H is certainly not embeddable into a group, and the chain of right ideals is given as

$$\begin{aligned} &H \supset xH \supset \cdots \leq x^n H \supset \cdots \supset yH \supset yxH \supset \cdots \supset y^n x^m H \supset \cdots \\ &\quad \supset \bigcup_{n \in \mathbb{N}} zx^{-n} H \supset \cdots \supset zx^{-n} H \supset \cdots \supset zH \leq zxH \supset \cdots \supset zx^n H \supset \cdots \\ &\quad \supset zyH \supset zyxH \supset \cdots \supset zy^m x^k H \supset \cdots \\ &\quad \supset \bigcup_{n \in N} z^2 x^{-n} H \supset \cdots \supset z^2 x^{-n} H \supset \cdots \supset z^2 H \supset z^2 xH \supset \cdots \\ &\quad \supset z^2 x^n H \supset \cdots \supset z^2 yH \supset z^2 yxH \supset \cdots \supset z^2 y^m x^k H \supset \cdots \supset \{\infty\}. \end{aligned}$$

To construct a group G that contains an invariant right cone $\mathscr{H}$ associated with H, we consider the wreath product G' of the infinite cyclic groups $\langle x_i \rangle$, $i \in Z$. To fix the notation, we recall that the base group C of the wreath product $A \wr B$ with A, B ordered groups is the direct sum $\sum_{b \in B} A_b$, where A_b is a copy of A for every $b \in B$. An element $c = (c_b)$, $c_b \in A_b$ is positive if $c_{b_0} > e$ in A_{b_0} for b_0 minimal with $c_b \neq e_{A_b}$. The wreath product $A \wr B = \{bc | b \in B,\ c \in C\}$ with $cb' = b'c^{b'}$ and $c_b^{b'} = c_{bb'^{-1}}$ is then the semidirect product of C and B, where the automorphisms induced by b' on C are the shifts as indicated. The group B can be embedded into $A \wr B$ by sending $b \in B$ to be_C and A is embedded into $A \wr B$ by mapping $a \in A$ to $e_B(c_b)$, where $c_{e_B} = a$ and $c_b = e_{A_b}$ otherwise. The group $A \wr B$ is also ordered with bc positive, if either b is positive in B or $b = e$ in B and c is positive in C.

We construct first the wreath product $U_{-1} = L_{-1}$ of the groups $\langle x_i \rangle$, $i = -1, -2, \ldots$ as a direct limit of

$$\langle x_{-1} \rangle, \langle x_{-2} \rangle \wr \langle x_{-1} \rangle, (\langle x_{-3} \rangle \wr \langle x_{-2} \rangle) \wr \langle x_{-1} \rangle, \text{etc.}$$

Then we consider

$$L_0 = L_{-1} \wr \langle x_0 \rangle \qquad \text{with subgroup } U_0 = \left\{ x_0^0 (w_i) | i \in \mathbb{Z}, w_i \in L_{-1} \right\}$$

and

$$L_1 = L_0 \wr \langle x_1 \rangle \qquad \text{with subgroup } U_1 = \left\{ x_1^0 (c_i) | i \in \mathbb{Z} \right\},$$

where

$$c_i = e_{L_0} \text{ for } i < 0,\ c_0 \in U_0, \text{ and } c_i \in L_0 \text{ for } i > 0.$$

If L_n, U_n has been defined, then $L_{n+1} = L_n \wr \langle x_{n+1} \rangle$ and $U_{n+1} = \{x^0_{n+1}(c_i),\ i \in \mathbb{Z}\}$ with $c_i = e_{L_n}$ for $i < 0$, $c_0 \in U_n$ and $c_i \in L_n$ for $i > 0$.

G' is then the direct limit of the L_n and contains the subgroup U, which is the direct limit of the U_n. The group G' is an ordered group, and it contains $x' = x_0$, $y' = x_1$ and $x' \cdot y' = y' \cdot u_1$ for some $u_1 \in U$, and $Ux' = x'U$ and $Uy' \subseteq y'U$. There is also an order-preserving automorphism σ of G' that maps x_i to x_{i-1}. Let G be the semidirect product $\langle z' \rangle \rtimes G'$, where $\langle z' \rangle$ is an infinite cyclic group and z' induces the automorphism σ on G'; i.e., $G = \{z'^n g' | n \in \mathbb{Z},\ g' \in G\}$ with $\{z'^{n_1} g')z'^{n_2} g'' = z'^{n_1+n_2} \sigma^{n_2}(g')g''$. Again G is an ordered group with positive cone $P = \{z'^n g' | n > 0 \text{ or } n = 0,\ g' \geq e_{G'}\}$. In G we define the set

$$\mathscr{H} = \left\{ z'^n y'^m x'^k u | n, m, k \geq 0,\ u \in U \right\} \cup \left\{ z'^n x'^{-k} u | n, k > 0,\ u \in U \right\}.$$

The mapping ϕ from H into $\mathscr{H}$ with $\phi(x) = x'$, $\phi(y) = y'$, $\phi(z) = z'$, $\phi(zx^{-n}) = z'x'^{-n}$ satisfies $\phi(h_1 h_2) = \phi(h_1)\phi(h_2) \cdot u$ for $h_1, h_2 \in H$ and suitable $u \in U$, since $x' \cdot z' = z'x_{-1}$, $x_{-1} \in U$ and $y'z' = z'x'$ and $Uz'^n x'^{-k} \subseteq z'^n x'^{-k} U$. It also follows that $\mathscr{H}$ is right invariant and associated with H. That $\mathscr{H}$ is associated with a right chain domain follows from Proposition 4.3.

If a right cone H is associated with a right cone $\mathscr{H}$ in an ordered group (G, P) (Problem 1), then we give an additional condition, such that H is associated to a right chain domain (Problem 2).

PROPOSITION 4.3. *Let (G, P) be an ordered group and H a right cone, $\{\infty\}$ a completely prime ideal, and $H \setminus \{\infty\}$ be contained in G such that $bH \subset aH$ for $a, b \in H \setminus \{\infty\}$ implies $a < b$. Then there exists a right chain domain R that is α-associated to H.*

Proof. We consider the Malcev–Neumann ring $D = Q[[G]]$ of all generalized power series with well-ordered support and define a subring R of D of all elements $\gamma = \sum h_i q_i$ with $\operatorname{supp}(\gamma) = \{h_i \in G | q_i \neq 0\} \subseteq H \setminus \{\infty\}$. Since $H \setminus \{\infty\}$ is a subsemigroup of G, it follows that R is indeed a ring.

Let $\gamma = \sum h_i q_i$ be an element in R and let h_0 be the minimal element in $S = \operatorname{supp}(\gamma)$. Then $h_0 < h_i$ for every $h_i \in S$, $h_i \neq h_0$, and $h_i H \subseteq h_0 H$ or $h_0 H \subset h_i H$. However, the second alternative implies $h_0 > h_i$, a contradiction, and $h_i = h_0 c_i$ for $c_i \in H$, $c_i > e$, follows.

Therefore, $\gamma = h_0 q_0 (1 + \sum c_i q_i') = h_0 q_0 (1 - m)$ with $m \in R$ such that every element in the support of m is positive. Hence, $(1 - m)^{-1} = \sum_{j=0}^{\infty} m^j$ exists in D and is contained in R. It follows that $\gamma R = h_0 R$, that

$h_1R = h_2R$ for $h_1, h_2 \in H$ holds if and only if $h_1H = h_2H$, since an element in R is a unit in R if and only if the minimal element in its support is a unit in H. Furthermore, $h_2R \subseteq h_1R$ if and only if $h_2H \subseteq h_1H$ for $h_1, h_2 \in H$. We show next that the extension of α with $\alpha(hH) = hR$ for all $h \in H \setminus \{\infty\}$ and $\alpha(\{\infty\}) = (0)$ to all right ideals is an association.

Let I be an ideal in H. Then $\alpha(I) = IR = \bigcup_{h \in I} hR$ is an ideal of R. Let $\gamma = \sum h_i q_i \in R$ and $hu \in \alpha(I)$ for $h \in I$, $u \in U(R)$. Then $\gamma \cdot h = \sum h_i q_i h = \sum h'_i q_i$ is an element in D, and every $h'_i = h_i h$ is in I. Hence, $\gamma hu \in \alpha(I)$. In addition, we have $\alpha(I) \cap H = I$.

Conversely, let $\hat{I}$ be an ideal of R and define $I = (\hat{I} \cap H)$. Then I is an ideal of H with $\hat{I} = \alpha(I) = IR$.

It follows that an ideal $I \subseteq H$ is completely prime if and only if the corresponding ideal $\alpha(I)$ is completely prime in R, since either ideal is completely prime if the square of an element is contained in it only if the element itself is contained in the ideal (see Lemma 1.10 (i)).

It remains to show that an ideal I of H is prime if and only if the corresponding ideal $\alpha(I)$ of R is prime. Assume that I is a prime ideal of H. Then $aHa \subseteq I$, $a \in H$ implies $a \in I$, and we want to show that $\gamma R \gamma \subseteq \alpha(I)$, $\gamma \in R$ implies $\gamma \in \alpha(I)$. Then $\gamma = h_0 q_0 + \dots$, where h_0 is the minimal element in the support of γ. We have $\gamma H \gamma \in \alpha(I)$ and hence $h_0 h h_0 \in I$ for every $h \in H$. Hence $h_0 \in I$ and $\gamma = h_0 u \in \alpha(I)$, where $u \in U(R)$.

Conversely, assume that $\alpha(I)$ is prime and that $h_1 H h_1 \subseteq I = \alpha(I) \cap H$. Let $\gamma = h_0 q_0 + \dots$ be an element in R, where h_0 is minimal in the support of γ. Then $h_1 \gamma h_1 = h_1 h_0 h_1 u$, $u \in U(R)$, is an element in $\alpha(I)$; hence $h_1 R h_1 \subseteq \alpha(I)$, and $h_1 \in I$ follows. ∎

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